The Signal, Variable System, and Transformation: A Personal Perspective

Shervin Erfani

315 Essex Hall

Faculty of Engineering

Outline Of the Talk

- **Introduction**
- **Mathematical Representation of systems**
- **Operator Calculus**
- **Transformation**
- **Observations On Laplace Transform**
- **SSB As An Example Of A Complex-Time System**
- **Observations On LTV Systems**
- **Time Variable**
- **A New Perspective**

Introduction

- **The classical theory of variable systems is based on the solutions of linear ordinary differential equations with varying coefficients. The varying coefficients are usually functions of an independent variable, also called the *time variable*. The *time variable* is assumed to be *real* for physical systems.**

$$\sum_{i=0}^{n} a_i(t)x_i^{(i)}(t) = \sum_{k=0}^{m} b_k(t)y_k^{(k)}(t)$$

What is an Operator Calculus?

The fundamental (differential) equation of an LTV system is:

$$\sum_{i=0}^{n} a_i \frac{d^i y}{dx^i} = f(x) \qquad a_n = 1$$

Use the operator $D^i \longrightarrow \dfrac{d^i}{dx^i}$

The fundamental equation converts to: $\displaystyle\sum_{i=0}^{n} a_i D^i y = f(x)$

Use the operator $s \longrightarrow D$

The fundamental equation (for a system at rest) converts to:

$$\sum_{i=0}^{n} a_i s^i y(x) = f(x)$$

Results of Using Operator Calculus

<u>Observation 1</u> – As a result of using the operator calculus the homogeneous response has a *pattern*. The response of homogeneous equation:

$$\sum_{i=0}^{n} a_i s^i y(x) = 0$$

is a linear combination of exponentials:

$$y(x) = \sum_{i=0}^{n} a_i e^{s_i x}$$

s_i's are roots of the operator (characteristic) equation:

$$\sum_{i=0}^{n} a_i s^i = 0$$

Extending the Operator Calculus: Transformation

Expand the fundamental equation:

$$\sum_{i=0}^{n} a_i \frac{d^i y}{dx^i} = f_n(x) = \sum_{i=0}^{n} f(i\Delta x)\Delta x$$

Assume an exponential solution (by generalization of the homogeneous solution):

$$y(x) = e^{s_n x}$$

The fundamental equation yields:

$$\sum_{i=0}^{n} a_i s_n^i e^{s_n x} = \sum_{i=0}^{n} f(i\Delta x)\Delta x$$

where s_n is a root of the operator (characteristic) equation:

$$\lim_{n \longrightarrow \infty} \sum_{i=0}^{n} a_i s_n^i = \lim_{n \longrightarrow \infty} \sum_{i=0}^{n} f(i\Delta x) e^{-s_n x} \Delta x$$

$$\mathcal{L}\{f(x)\} = F(s) = \int_0^{+\infty} f(x) e^{-sx} dx$$

Laplace Transform: Good or Bad?

Introduced by Laplace in 1771 and applied (modern use) by Oliver Heaviside.

Observation 2 – The Laplace transform is obtained as a result of extending the concept of the operator calculus for solving differential equations, which can describe the fundamental equation of physical (dynamic) systems.

Observation 3 – The solution exists if there are finite numbers *M* and σ_0 such that :

$$\left| f(x) \right| < M e^{\sigma_0 x} \qquad \forall x \geq 0$$

.Observation 4 – The independent variable x can represent any parameter (of the system); e.g., the "time."

Observation 5 – s is the root of the characteristic equation $F(s) = 0$

Hence it is a complex number (or better said, a complex variable) in general.

Observation 6 – If x represents the independent time variable, then by definition, s represents the (complex) frequency in the *transform domain*.

Question 1 – Can f(x) be a complex function of x?

Question 2 – Can x represent an independent complex variable?

Complex Time System?

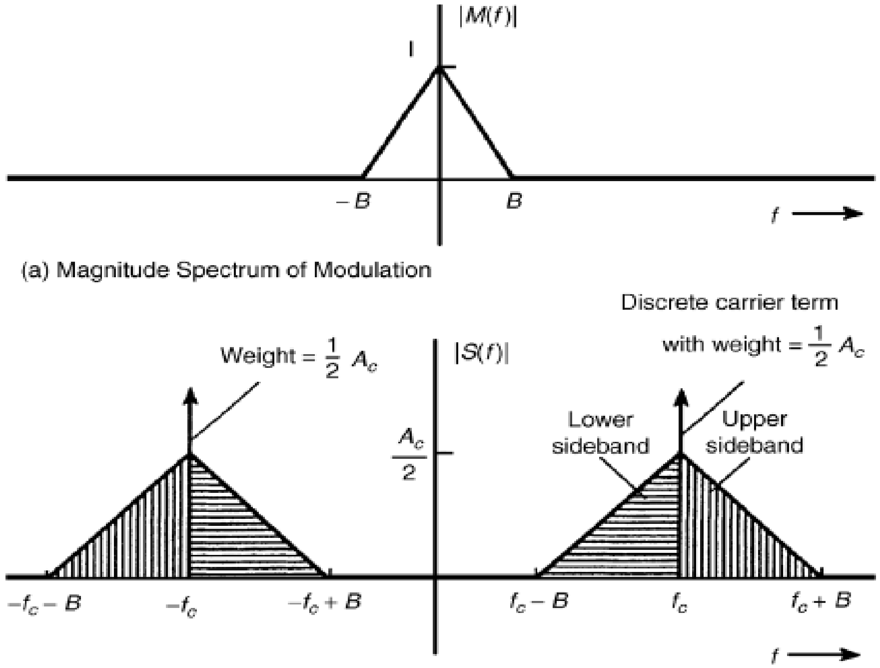
The single side band (SSB) amplitude modulation (AM) is an example of "complex-time" systems. The SSB spectrum is obtained by shifting the spectra

$$M_{+}(\omega) = M(\omega)u(\omega)$$

and

$$M_{-}(\omega) = M(\omega)u(-\omega)$$

by ω_c and $-\omega_c$ respectively, as shown.



(a) Magnitude Spectrum of Modulation

(b) Magnitude Spectrum of AM Signal

SSB System (Cont.)

It can be shown that:

$$m_+(t) = \frac{1}{2}\left[m(t) + jm_h(t)\right]$$

$$m_-(t) = \frac{1}{2}\left[m(t) - jm_h(t)\right]$$

:

where $m_h(t)$ is the Hilbert transform of $m(t)$

$$m_h(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{m(\tau)}{t-\tau}d\tau$$

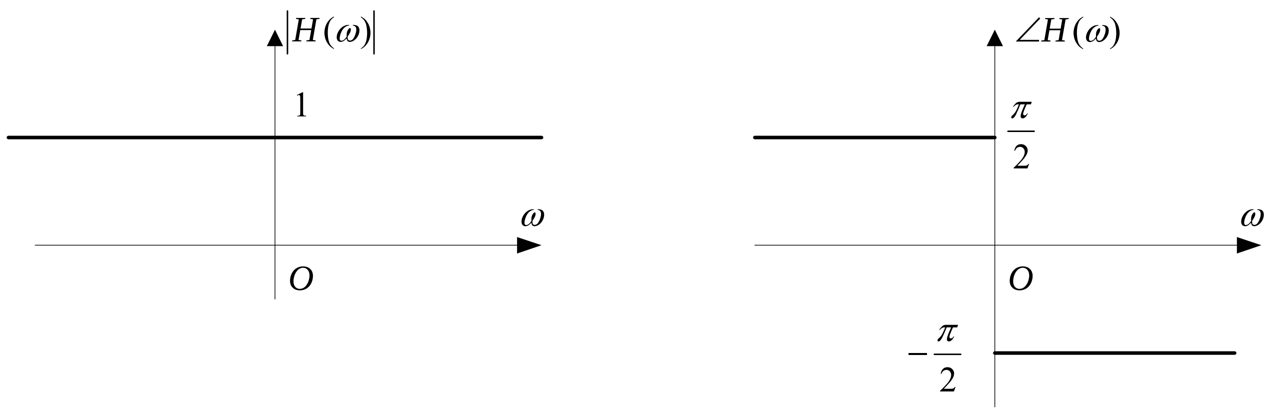
Fourier transform of $m_h(t)$ is: $M_h(\omega) = -jM(\omega)\operatorname{sgn}(\omega) = M(\omega)H(\omega)$ where:

$$H(\omega) \stackrel{\Delta}{=} -j\operatorname{sgn}(\omega) = \begin{cases} -j = e^{-j\frac{\pi}{2}} & \omega > 0 \\ j = e^{j\frac{\pi}{2}} & \omega < 0 \end{cases}$$

$H(\omega)$ is the transfer function of $m_+(t)$ an ideal $\frac{\pi}{2}$

phase shifter that produces the imaginary-time part of the real-time function $m_+(t)$

SSB System (cont.)



Representation of the transfer function $H(\omega)$, an ideal $\frac{\pi}{2}$ phase-shifter.

The USB signal is:

$$\Phi_{USB}(\omega) = M_+(\omega - \omega_c) + M_-(\omega + \omega_c)$$

In time domain:

$$\varphi_{USB}(t) = m_+(t)e^{j\omega_c t} + m_-(t)e^{-j\omega_c t}$$

Substituting for $m_+(t)$ and $m_-(t)$ in this equation results in:

$$\varphi_{USB}(t) = m(t)\cos\omega_c t - m_h(t)\sin\omega_c t$$

Example of an SSB System

For $M(\omega) = 2\pi e^{-a|\omega|}$ find $m_+(t)$.

$$\mathcal{F}^{-1}\{M(\omega)\} = \mathcal{F}^{-1}\{2\pi e^{-a|\omega|}\} = \frac{2a}{t^2 + a^2}$$

The Fourier transform of the Hilbert transform of $M(\omega)$ is:

$$M_h(\omega) = -jM(\omega)\operatorname{sgn}(\omega) = -j2\pi\left[e^{-a\omega}u(\omega) - e^{a\omega}u(-\omega)\right]$$

The Hilbert transform is:

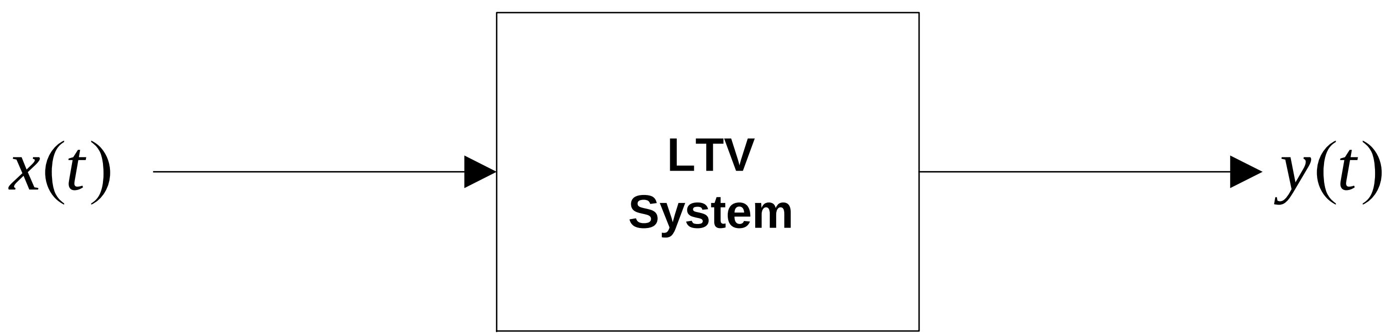
$$m_h(t) = \mathcal{F}^{-1}\{M_h(\omega)\} = -j\left[\frac{1}{a - jt} - \frac{1}{a + jt}\right] = \frac{2t}{t^2 + a^2}$$

$$m_+(t) = \frac{1}{2}\left[m(t) + jm_h(t)\right] = \frac{a + jt}{t^2 + a^2}$$

Characterization of LTV systems

- **Consider a single-input single-output (SISO) linear dynamic system characterized by the fundamental (differential) equation of an LTV system:**

$$\sum_{i=0}^{n} a_i(t) x_i^{(i)}(t) = \sum_{k=0}^{m} b_k(t) y_k^{(k)}(t)$$



Characterization of LTV systems (Cont.)

Characterization in Operator form:

$$\sum_{i=0}^{n} a_i(t) D^i y(t) = \sum_{i=0}^{n} b_i(t) D^i x(t)$$

$$L(D,t)y(t) = K(D,t)x(t)$$

where:

$y(t)$ **= the output response signal**

$x(t)$ **= the input (excitation) signal**

$a_i(t)$ **= system variable parameter, known continuous function of time**

$b_k(t)$ **= system time-varying parameter, known continuous function of time**

D_i **= the ith differential operator (**d^i / dt_i**)**

$L(\cdot,\cdot)$**= the system output operator, known bivariate polynomial of time and differential operator**

$K(\cdot,\cdot)$**= the system input operator, known bivariate polynomial of time and differential operator**

Observations on the LTV systems

Observation 1 – In general, time clocks of the signal and system are not synchronized; i.e., the (time) variables of the signal and systems are *independent* of each other.

$$L(D,\tau)y(t) = K(D,\tau)x(t)$$

Observation 2 – At any instant of "t" there is a response, which is a specified function of "τ".

Observation 3 – At any fixed "τ" there is a response, which is a specified function of "t".

Observation 4 – The system response is a function of variations of observation parameter "t" and application parameter "τ".

Observation 5 – A zero-input, SISO LTV system described by:

$$L(D,\tau)y(\cdot) = 0$$

is a linear system that its natural frequencies are varying with "τ". In other words, the solutions of this equation are exponential functions of time with varying natural frequencies, as given by:

$$y(\cdot) = \sum_{i=0}^{n} c_i e^{-t\alpha_i(\tau)}$$

where $\alpha_i(\tau)$ is a function of variable coefficients of the fundamental equation of the system under consideration.

Extension of the Operator Calculus to Solution of LTV Systems

Observation 6 – Considering the invariance property of

$$L(D,\tau)y(t) = K(D,\tau)x(t)$$

with respect to "t" and "τ", and by analogy with the case of LTI systems, we interpret this equation as a two-dimensional system model, and shall use a two-dimensional operator calculus (i.e. two-dimensional Laplace transform (2DLT)) to find its response.

$$\mathcal{L}_{2D}\{L(D,\tau)y(t)\} = \mathcal{L}_{2D}\{K(D,\tau)x(t)\}$$

$$L(s_1,s_2)Y(s_1) = K(s_1,s_2)X(s_1)$$

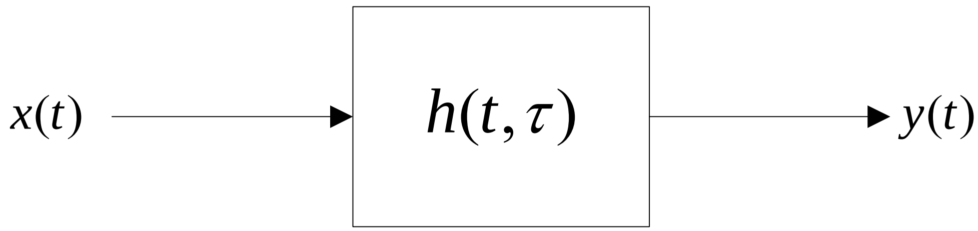
$$Y(s_1) = \frac{K(s_1,s_2)}{L(s_1,s_2)}X(s_1)$$

where:

$$K(s_1,s_2) = \sum_{i=0}^{n} B_i(s_2)s_1^i$$

$$L(s_1,s_2) = \sum_{i=0}^{n} A_i(s_2)s_1^i$$

2DLT Solution of LTV Systems



If $x(t,\tau)=\delta(t,\tau)=\delta(t)\delta(\tau)$ we denote the 2D transfer function, $H(s_1,s_2)$ of an LTV system as:

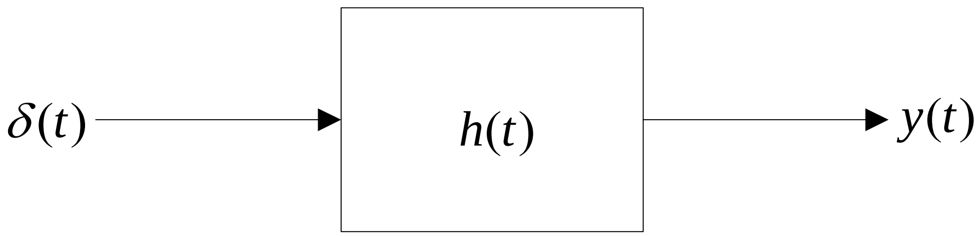
$$H(s_1,s_2)=\frac{K(s_1,s_2)}{L(s_1,s_2)}=\sum_{i=0}^{n}\frac{B_i(s_2)s_1^i}{A_i(s_2)s_1^i}$$

Where:

$$h(t,\tau)=\mathbf{L}^{-1}_{\mathbf{2D}}\{H(s_1,s_2)\}=\frac{1}{(2\pi j)^2}\int_{\sigma_2-j\infty}^{\sigma_2+j\infty}\int_{\sigma_1-j\infty}^{\sigma_1+j\infty}H(s_1,s_2)e^{s_1t}e^{s_2\tau}ds_1ds_2$$

$H(s_1,s_2)$ and $h(t,\tau)$ are called the bi-frequency transfer function and bivariate impulse response, respectively.

Special Case: LTI System



In the case of LTI systems, coefficients a_i and b_i are all constants; $H(s_1, s_2)$ reduces to the familiar transfer function

$$H(s) = \frac{K(s)}{L(s)} = \sum_{i=0}^{n} \frac{b_i s^i}{a_i s^i}$$

Note that letting $s_1 = s_2 = j\omega$

results in the inverse of the two-dimensional Fourier transform (2DFT) of $H(j\omega, j\omega)$

as given by:

$$h(t,\tau) = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} H(j\omega, j\omega) e^{-j\omega(t+\tau)} d\omega d\omega$$

A Word on *Time Variable*

- The "time variable" is assumed to be a *real* variable for physical systems. This assumption facilitates analysis and synthesis of fixed (time-invariant) systems by allowing the *Laplace transform* techniques to be used.
- However, the assumption of "real time" is shown to be inadequate for realization of time-varying systems in the *transform domain.*

A New Perspective

- **The discussion in this presentation is based on a different point of view.**
- **Possibility of system realization through an examination of the behavior of systems that are functions of a *complex* time-variable.**
- **This approach allows, in effect, a two-dimensional Laplace transform (2DLT) technique to be used for the time-varying systems in the same manner that the conventional frequency-domain techniques are used in connection with fixed systems.**

To Be Continued …

Or

The End